# HERMITE-HADAMARD'S INEQUALITIES FOR PREQUASIINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions via fractional integrals.

## 1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping. For several recent results concerning the inequality (1.1) we refer the interested reader to [1, 2, 3, 6].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if inequality

$$f\left(tx + (1-t)y\right) \leq \max\left\{f(x), f(y)\right\},$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Furthemore there exist quasi-convex functions which are not convex (see [5]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.** *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

*and*

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$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x<b$$

*respectively, where* $\Gamma(\alpha)$ *is the Gamma function and* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent result connected with fractional integral inequalities see ([6]-[9]).

In [9], Ozdemir and Yıldız proved the Hadamard inequality for quasi-convex functions via Riemann-Liouville fractional integrals as follows:

**Theorem 1.** *Let* $f:[a,b]\to\mathbb{R}$, *be positive function with* $0\le a<b$ *and* $f\in L[a,b]$. *If* $f$ *is a quasi-convex function on* $[a,b]$, *then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \le \max\left\{f(a), f(b)\right\} \tag{1.2}$$

*with* $\alpha>0$.

**Theorem 2.** *Let* $f:[a,b]\to\mathbb{R}$, *be a differentiable mapping on* $(a,b)$ *with* $a<b$. *If* $|f'|$ *is quasi-convex on* $[a,b]$, $\alpha>0$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}(1.3)\qquad &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\le \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{|f'(a)|, |f'(b)|\right\}.\end{aligned}$$

**Theorem 3.** *Let* $f:[a,b]\to\mathbb{R}$, *be a differentiable mapping on* $(a,b)$ *with* $a<b$ *such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is quasi-convex on* $[a,b]$, *and* $q>1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}(1.4)\qquad &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\le \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha\in[0,1]$.

**Theorem 4.** *Let* $f:[a,b]\to\mathbb{R}$, *be a differentiable mapping on* $(a,b)$ *with* $a<b$ *such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is quasi-convex on* $[a,b]$, *and* $q\ge 1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}(1.5)\qquad &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\le \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

*with* $\alpha>0$.

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [10]. Weir and Mond [11] introduced the

concept of preinvex functions and applied it to the establisment of the sufficient optimality conditions and duality in nonlinear programming. Pini [12] introduced the concept of prequasiinvex function as a generalization of invex functions. Later, Mohan and Neogy [20] obtained some properties of generalized preinvex functions. Noor [13]-[15] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions.In recent papers Yang et al.in [18] studied prequasiinvex functions, and semistrictly prequasiinvex functions and Barani et al. in [19] presented some generaliztions of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions.

In this paper we generalized the results in [9] for prequasiinvex functions. Now we recall some notions in invexity analysis which will be used throught the paper (see [16, 17] and references therein)

Let $f : A \to \mathbb{R}$ and $\eta : A \times A \to \mathbb{R}$,where $A$ is a nonempty set in $\mathbb{R}^n$, be continuous functions.

**Definition 2.** *The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$,*

$$x + t\eta(y, x) \in A.$$

*The invex set $A$ is also called a $\eta-$connected set.*

It is obvious that every convex set is invex with respect to $\eta(y, x) = y - x$, but there exist invex sets which are not convex [16].

**Definition 3.** *The function $f$ on the invex set $A$ is said to be preinvex with respect to $\eta$ if*

$$f(x + t\eta(y, x)) \leq (1 - t) f(x) + t f(y), \ \forall x, y \in A, \ t \in [0, 1].$$

*The function $f$ is said to be preconcave if and only if $-f$ is preinvex.*

**Definition 4.** *The function $f$ on the invex set $A$ is said to be prequasiinvex with respect to $\eta$ if*

$$f(x + t\eta(y, x)) \leq \max\{f(x), f(y)\}, \ \forall x, y \in A, \ t \in [0, 1].$$

Every quasi-convex function is a prequasinvex with respect to $\eta(y, x) = y - x$, but the converse does not holds (see example 1.1 in [18])

We also need the following assumption regarding the function $\eta$ which is due to Mohan and Neogy [20]:

**Condition C:** Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. For any $x, y \in A$ and any $t \in [0, 1]$,

$$\begin{aligned} \eta(y, y + t\eta(x, y)) &= -t\eta(x, y) \\ \eta(x, y + t\eta(x, y)) &= (1 - t)\eta(x, y). \end{aligned}$$

Note that for every $x, y \in A$ and every $t \in [0, 1]$ from condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y). \tag{1.6}$$

In [19] Barani et al. proved the Hermite-Hadamard type inequality for prequasiinvex as follows:

**Theorem 5.** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on $A$ then, for every $a, b \in A$ the following inequalities holds*

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ (1.7) \qquad \leq \; & \frac{|\eta(b,a)|}{4} \max\left\{|f'(a)|, |f'(b)|\right\}. \end{aligned}$$

**Theorem 6.** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on $A$ then, for every $a, b \in A$ the following inequalities holds*

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ (1.8) \qquad \leq \; & \frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}} \left( \max\left\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \end{aligned}$$

In [21], Iscan proved the following Lemma and established some inequalities for preinvexfunctions via fractional integrals

**Lemma 1.** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $f'$ is preinvex function on $A$ and $f' \in L[a, a + \eta(b,a)]$ then, the following equality holds:*

$$\begin{aligned} (1.9) \qquad & \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left[ J^{\alpha}_{a^+} f(a + \eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a) \right] \\ = \; & \frac{\eta(b,a)}{2} \int_0^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] f'(a + t\eta(b,a))\, dt \end{aligned}$$

In this paper, using lemma 1 we obtained new inequalities related to the right side of Hermite-Hadamard inequalities for prequasiinvex functions via fractional integrals.

## 2. Main Results

**Theorem 7.** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. If $f : [a, a + \eta(b,a)] \to (0, \infty)$ is a prequasiinvex function, $f \in L[a, a + \eta(b,a)]$ and $\eta$ satisfies condition C then, the following inequalities for fractional integrals holds:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left[ J^{\alpha}_{a^+} f(a + \eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a) \right] \\ (2.1) \qquad \leq \; & \max\{f(a), f(a + \eta(b,a)\} \leq \max\{f(a), f(b)\} \end{aligned}$$

*Proof.* Since $a, b \in A$ and $A$ is an invex set with respect to $\eta$, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in A$. By prequasiinvexity of $f$ and inequality (1.6) for every

$t \in [0,1]$ we get

$$\begin{aligned} f(a+t\eta(b,a)) &= f(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a), f(a+\eta(b,a)\} \end{aligned} \tag{2.2}$$

and similarly

$$\begin{aligned} f(a+(1-t)\eta(b,a)) &= f(a+\eta(b,a)+t\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a), f(a+\eta(b,a)\}. \end{aligned}$$

By adding these inequalities we have

$$f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a)) \leq 2\max\{f(a), f(a+\eta(b,a)\} \tag{2.3}$$

Then multiplying both (2.3) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} f(a+t\eta(b,a))\,dt + \int_0^1 t^{\alpha-1} f(a+(1-t)\eta(b,a))\,dt \leq 2\max\{f(a), f(a+\eta(b,a)\} \int_0^1 t^{\alpha-1}dt.$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right] \leq \frac{2\max\{f(a), f(a+\eta(b,a)\}}{\alpha}.$$

Using the mapping $\eta$ satisfies condition C the proof is completed. □

**Remark 1.** *If in Theorem 7, we let $\eta(b,a) = b-a$, then inequality (2.1) become inequality (1.2) of Theorem 1.*

**Theorem 8.** *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a+\eta(b,a)$ such that $f' \in L[a, a+\eta(b,a)]$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex function on $A$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right]\right| \\ &\leq \frac{\eta(b,a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\{|f'(a)|, |f'(b)|\}. \end{aligned} \tag{2.4}$$

*Proof.* Using lemma 1 and the prequasiinvexity of $|f'|$ we get

$$\begin{aligned}
& \left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
\leq & \frac{\eta(b,a)}{2}\left\{\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt\right\} \\
= & \eta(b,a)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt\right) \\
= & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\},
\end{aligned}$$

which completes the proof. ☐

**Remark 2.** *a) If in Theorem 8, we let $\eta(b,a)=b-a$, then inequality (2.4) become inequality (1.3) of Theorem 2*

*b) If in Theorem8, we let $\alpha=1$, then inequality (2.4) become inequality (1.7) of Theorem 5.*

*c) In Theorem8, assume that $\eta$ satisfies condition C.Using inequality (2.2) we get*

$$\begin{aligned}
& \left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(a+\eta(b,a))\right|\right\}
\end{aligned}$$

**Theorem 9.** *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|^{q}$ is prequasiinvex function on $A$ for some fixed $q>1$ then the following inequality holds:*

$$\begin{aligned}
(2.5)\quad & \left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

*where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.*

*Proof.* From lemma1 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\;& \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|^{\frac{1}{p}+\frac{1}{q}}\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq\;& \frac{\eta(b,a)}{2}\left(\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|dt &= \int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).
\end{aligned}$$

Since $|f'|^{q}$ is prequasiinvex function on $A$, we obtain

$$\left|f'\left(a+t\eta(b,a)\right)\right|^{q}\leq\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\},\quad t\in[0,1]$$

and

$$\begin{aligned}
\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt &\leq \int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}dt \\
&= \max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\right) \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}
\end{aligned}$$

from here we obtain inequality (2.5) which completes the proof. □

**Remark 3.** *a) If in Theorem9, we let $\eta(b,a)=b-a$ then inequality (2.5)become inequality (1.5) Theorem4.*

*b) In Theorem9, assume that $\eta$ satisfies condition C.Using inequality (2.2) we get*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\;& \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 10.** *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that*

*$f : A \to \mathbb{R}$ is a differentiable function. If $|f'|^q$ is prequasiinvex function on $A$ for some fixed $q > 1$ then the following inequality holds:*

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right| \tag{2.6}$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q,|f'(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}.$$

*where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha \in [0,1]$.*

*Proof.* From lemma1 and using Hölder inequality with properties of modulus, we have

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1 |t^{\alpha}-(1-t)^{\alpha}|\,|f'\left(a+t\eta(b,a)\right)|\,dt$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 |t^{\alpha}-(1-t)^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'\left(a+t\eta(b,a)\right)|^q\,dt\right)^{\frac{1}{q}}.$$

We know that for $\alpha \in [0,1]$ and $\forall t_1, t_2 \in [0,1]$,

$$|t_1^{\alpha}-t_2^{\alpha}| \leq |t_1-t_2|^{\alpha},$$

therefore

$$\begin{aligned}\int_0^1 |t^{\alpha}-(1-t)^{\alpha}|^p\,dt &\leq \int_0^1 |1-2t|^{\alpha p}\,dt \\ &= \int_0^{\frac{1}{2}} [1-2t]^{\alpha p}\,dt+\int_{\frac{1}{2}}^1 [2t-1]^{\alpha p}\,dt \\ &= \frac{1}{\alpha p+1}.\end{aligned}$$

Since $|f'|^q$ is prequasiinvex on $[a, a+\eta(b,a)]$, we have inequality (2.6), which completes the proof. $\square$

**Remark 4.** *a) If in Theorem 10, we let $\eta(b,a)=b-a$ then inequality (2.6) become inequality (1.4) of Theorem3.*

*b) If in Theorem 10, we let $\alpha = 1$then inequality (2.6) become inequality (1.8) of Theorem6.*

*c) In Theorem 10, assume that $\eta$ satisfies condition C. Using inequality (2.2) we get*

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q,|f'(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}.$$

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